Output Feedback Invariants*

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Abstract

The paper is concerned with the problem of determining a complete set of invariants for output feedback. Using tools from geometric invariant theory it is shown that there exists a quasi-projective variety whose points parameterize the output feedback orbits in a unique way. If the McMillan degree $n \geq mp$, the product of number of inputs and number of outputs, then it is shown that in the closure of every feedback orbit there is exactly one nondegenerate system.

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1 Introduction

Consider a time invariant linear (strictly proper) system

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1.1}$$

having m inputs, p outputs, and n states. The (full) feedback group is the group generated through the feedback action:

$$u \longmapsto u + Fy \tag{1.2}$$

and through the change of basis in state space, input space and output space respectively, i.e. through the transformations:

$$\begin{aligned} x &\longmapsto Sx, && S \in Gl_n && (1.3)\\ u &\longmapsto T_1 u, && T_1 \in Gl_m && (1.4)\\ y &\longmapsto T_2 y, && T_2 \in Gl_p. && (1.5) \end{aligned}$$

The orbits under the full feedback group are referred to as the *output feedback orbits*. In order to fully understand the effect of output feedback on the structure of linear systems it is of fundamental interest to (i) classify the feedback orbits, (ii) to determine a complete set invariants for output feedback and (iii) to obtain a detailed description of the adherence order (orbit closure inclusion) of the different orbits.

Those obviously important problems have already been studied by many authors (see e.g. [3, 4, 5, 6, 7, 8, 17]) and despite many partial results the problem is still far from being solved.

The transformations induced by the actions (1.2), (1.3), (1.4), (1.5) describe a group action on the vector space of all matrix triples (A, B, C) which is a vector space of dimension $n(m + n + p)$. There is an extensive mathematical literature on the classification of orbits arising from group actions on vector spaces and more general algebraic varieties and we refer to Section 3 for some more details. If the number of orbits is finite then this study generally seeks a discrete set of invariants classifying the finitely many orbits. There are a few instances in the control literature where the set of orbits is finite and as examples we refer to [2, 8, 9].

In the problem at hand the number of feedback orbits is in general infinite and this makes the problem difficult. In order to classify all orbits it will therefore be necessary to derive a 'continuous set of invariants'.

The application of tools from geometric invariant theory (see e.g. [13, 14]) often enables one to derive for a given group action a set of invariants in a systematic way.

From a geometric point of view this amounts to describing an algebraic variety whose points parameterize uniquely the closed feedback orbits.

In this paper we construct, using tools from geometric invariant theory, such a quasi-projective algebraic variety, whose points parameterize closed output feedback orbits in a unique way. Since a quasi-projective variety can be embedded into affine space using e.g. semi-algebraic functions our result implies the existence of a complete set of semi-algebraic invariants for output feedback. It also helps to construct such a complete set of invariants; however this problem will not be addressed here.

In order to achieve the result it is crucial to first extend the output feedback action to an action that operates on a compactification of the space of proper transfer functions of McMillan degree n. This process will be explained in Section 2. In Section 3 we summarize some important notions from geometric invariant theory to the extend we will need it in this paper.

In order to apply the theorems from geometric invariant theory to the output feedback invariant problem it will be necessary to compactify the manifold of $p \times m$ transfer functions of McMillan degree n. This will be accomplished in Section 4 using the so called space of homogeneous autoregressive systems [15].

The main results of the paper are provided in Section 5, where we show that the space of homogeneous autoregressive systems contains a non-empty Zariski open subset of semi-stable orbits. This in turn will then lead to a quasi-projective variety which parameterizes the set of output feedback invariants in a continuous manner.

In Section 6 we reinterpret the obtained results in terms of generalized first order representations. Finally in Section 7 we concretely describe the quasi-projective variety derived in Section 5 in the situation of single output systems.

2 Cascade equivalence and the extended feedback group

The notion of *cascade equivalence* was introduced by Byrnes and Helton in [4] and it is closely related to the feedback classification problem. In our context this notion can be equivalently described in the following way:

Consider a time invariant linear **proper** system of the form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du. \tag{2.1}$$

In addition to the feedback action (1.2) and the basis transformations (1.3), (1.4)

and (1.5) we will also allow a feed-forward transformation

$$y \longmapsto y + Gu. \tag{2.2}$$

The collection of all those transformations will be called the *extended full feedback group*. The actions in (1.2), (1.4), (1.5) and (2.2) describe invertible transformations on the space of external variables $[u^t \; y^t]^t$. In this way we can view these actions as elements of the general linear group $T \in Gl_{m+p}$ and the collection of these transformations is compactly described through:

$$\begin{bmatrix} u \\ y \end{bmatrix} \longmapsto \begin{bmatrix} T_1 & F \\ G & T_2 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = T \begin{bmatrix} u \\ y \end{bmatrix}, \tag{2.3}$$

where $T \in Gl_{m+p}$. This shows that (1.2), (1.4), (1.5) and (2.2) generate the whole general linear group $T \in Gl_{m+p}$.

Note that the linear transformation $T \in Gl_{m+p}$ induces the notion of cascade equivalence on the set of proper transfer functions. The following Lemma is easily established:

Lemma 2.1. *There is a bijective correspondence between the set of equivalence classes of the form* (2.1) *under the extended full feedback group and the set of equivalence classes of the form* (1.1) *under the full feedback group.*

This Lemma now enables us to concentrate on the linear transformation (2.3). Instead of working with a state space description we can also work with polynomial matrices. For this let

$$D^{-1}(s)N(s) := G(s) := C(sI - A)^{-1}B + D$$

be a left coprime factorization of the transfer function of system (2.1). Then the linear transformation (2.3) is equivalently described through:

$$(D(s) \; N(s)) \longmapsto (D(s) \; N(s))\, T^{-1} \quad T \in Gl_{m+p}. \tag{2.4}$$

3 Basic notions from geometric invariant theory

Geometric invariant theory constitutes an active research area of algebraic geometry. One of the main references is the book by Mumford and Fogarty [13]. The non-specialists among the interested readers will find the book by Newstead [14] a good introductory book.

In this section we explain an important result from geometric invariant theory which we will use later in the paper to derive a set of continuous feedback invariants.

Let X be a projective variety, i.e. X is the zero locus of a finite set of homogeneous polynomial equations. We will assume that X is embedded into the projective space $\mathbb{P}^N$. Let $G \subset Gl_{N+1}$ be a reductive group (such as, e.g., a group isomorphic to the general linear group) which acts on the projective space $\mathbb{P}^N$ and induces an action on the variety X.

In this situation one has the following general result: (see [14, Theorem 3.14]).

Theorem 3.1. *There exists a Zariski open set X^{ss} of so called* semi-stable *points, a projective variety Y and an algebraic morphism $\phi : X^{ss} \to Y$ having the property that $\phi^{-1}(y)$ contains exactly one closed G orbit for every $y \in Y$. Moreover there is a Zariski open set $Y^s \subset Y$ such that $\phi^{-1}(y)$ contains one and only one orbit for every $y \in Y^s$.*

The set $X^s := \phi^{-1}(Y^s)$ is the so called set of stable points and both X^s and X^{ss} are Zariski open sets of the variety X. It is possible that X^{ss} is the empty set in which case Theorem 3.1 does not give any insight.

Theorem 3.1 is significant in several ways. First the points of the variety Y provide a continuous family of invariants capable of distinguishing orbits inside X^s. In addition the variety Y is characterized through some universal properties. Because of this reason one sometimes also speaks about the *categorical quotient* Y.

The fact that this variety Y is projective is surprising. It will be our goal in the next section to apply Theorem 3.1 to the feedback orbit classification problem.

4 The projective variety of homogeneous autoregressive systems

In order to apply Theorem 3.1 it will therefore be necessary to compactify the space of all $p \times (m+p)$ autoregressive systems of the form $P(s) = (D(s)\ N(s))$. Such a compactification was provided in [15] and we shortly review the details.

Consider a $p \times (m+p)$ polynomial matrix

$$P(s,t) := \begin{pmatrix} f_{1,1}(s,t) & \dots & f_{1,(m+p)}(s,t) \\ f_{2,1}(s,t) & \dots & f_{2,(m+p)}(s,t) \\ \vdots & & \vdots \\ f_{p,1}(s,t) & \dots & f_{p,(m+p)}(s,t) \end{pmatrix}. \tag{4.1}$$

We say $P(s,t)$ is homogeneous of row degrees $\nu_1, \ldots, \nu_p$ if each element $f_{i,j}(s,t)$ is a homogeneous polynomial of degree ν_i. A square matrix $U(s,t)$ of homogeneous polynomials is called unimodular, if $\det U(s,t)$ is a nonzero monomial in t. We say two homogeneous matrices $P(s,t)$ and $\tilde{P}(s,t)$ are equivalent if they have the same row-degrees and if there is a unimodular matrix $U(s,t)$, whose entries $u_{ij}(s,t)$ are homogeneous polynomials of degree $\nu_i - \nu_j$ with $\tilde{P}(s,t) = U(s,t)P(s,t)$. Using this equivalence relation we define:

Definition 4.1. An equivalence class of full rank homogeneous polynomial matrices $P(s,t)$ will be called a *homogeneous autoregressive system*. The McMillan degree of a homogeneous autoregressive system is defined as the sum of the row degrees, i.e. through $n := \sum_{i=1}^{p} \nu_i$. The set of all homogeneous autoregressive systems of size $p \times (m+p)$ and McMillan degree n will be denoted by $\mathcal{H}^n_{p,m}$.

Let $\mathcal{R}_{n,m,p}$ denote the space of $p \times m$ proper transfer functions of McMillan degree n. The main result established in [15] is as follows:

Theorem 4.2 ([15]). *$\mathcal{H}^n_{p,m}$ is a smooth projective variety containing the set of proper transfer functions $\mathcal{R}_{n,m,p}$ as a Zariski dense subset.*

More generally, $\mathcal{H}^n_{p,m}$ contains the set of all degree n rational curves of the Grassmannian Grass $(m, m+p)$ as a Zariski-dense subset. The variety $\mathcal{H}^n_{p,m}$ arises in algebraic geometry in the following context: Let $\mathbb{P}^1$ be the projective line and let $O_{\mathbb{P}^1}$ be the structure sheaf of $\mathbb{P}^1$. Let V be an $(m+p)$-dimensional vector space over the base field K. The space $\mathcal{H}^n_{p,m}$ is the quotient scheme that parameterizes all quotients $\mathcal{B}$ of the sheaf $V \otimes O_{\mathbb{P}^1}$ of degree of n and rank m, that is, sheaves $\mathcal{B}$ whose Hilbert polynomial is $\chi(\ell) = m(\ell+1)+n$. This identification proceeds as follows(refer to [15] for more details): A point x in the quotient scheme gives rise to a short exact sequence:

$$0 \to \mathcal{A} \xrightarrow{\psi} V \otimes O_{\mathbb{P}^1} \xrightarrow{\phi} \mathcal{B} \to 0 \tag{4.2}$$

Now $\mathcal{A}$ is a locally free sheaf of degree $-n$ and rank p. So $\mathcal{A} \simeq \bigoplus_{i=1}^{l} O_{\mathbb{P}^1}(-\nu_i)$. Therefore the map from $\mathcal{A}$ to $V \otimes O_{\mathbb{P}^1}$ is given by the transpose of a homogeneous autoregressive system of the form $P(s,t)$, once a basis is chosen for V and $\mathcal{A}$. Conversely given a homogeneous autoregressive system $P(s,t)$ the map defined by the transpose of $P(s,t)$ from $\bigoplus_{i=1}^{l} O_{\mathbb{P}^1}(-\nu_i)$ to $V \otimes O_{\mathbb{P}^1}$ is an injective map since P has full rank. The quotient of this map has rank m and degree n and thus defines a point x in the quotient scheme.

The space $\mathcal{H}^n_{p,m}$ can be embedded as a projective variety in the following way: Fix an integer $\ell \geq n$. Given a point $x \in \mathcal{H}^n_{p,m}$ corresponding to the short exact sequence of sheaves (4.2) one has the corresponding long exact sequence of cohomology groups:

$$0 \to H^0(\mathbb{P}^1, \mathcal{A}(\ell)) \to H^0(\mathbb{P}^1, V \otimes O_{\mathbb{P}^1}(\ell)) \to H^0(\mathbb{P}^1, \mathcal{B}(\ell)) \to H^1(\mathbb{P}^1, \mathcal{A}(\ell)) \to \cdots$$

Now, if $\ell \geq n$ then $H^1(\mathbb{P}^1, \mathcal{A}(\ell)) = 0$. Also $\dim H^0(\mathbb{P}^1, \mathcal{A}(\ell)) = p(\ell+1) - n$ and $H^0(\mathbb{P}^1, V \otimes O_{\mathbb{P}^1}(\ell)) = (m+p)(\ell+1) \simeq V \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\ell))$, so that $\dim H^0(\mathbb{P}^1, V \otimes O_{\mathbb{P}^1}(\ell)) = (m+p)(\ell+1)$.

Finally one obtains a map $\rho_\ell : \mathcal{H}^n_{p,m} \to \mathrm{Grass}(p(\ell+1)-n, V \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\ell))$, the Grassmanian of $p(\ell+1)-n$)-dimensional subspaces of $V \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\ell)$, obtained by defining $\rho_\ell(x)$ to be the subspace $H^0(\mathbb{P}^1, \mathcal{A}(\ell))$. The map ρ_ℓ defines an embedding (see [15] where it is proved specifically for $\ell = n$, but the same proof applies to $\ell > n$). This Grassmannian can be embedded through the Plücker embedding in $\mathbb{P} = \mathbb{P}(\overset{(p(\ell+1)-n)}{\wedge} V \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\ell)))$, the projective space of lines in this vector space. The group $\mathrm{Gl}(V)$ obviously acts on V, therefore also on $V \otimes O_{\mathbb{P}^1}$ and thus also on the vector space $V \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\ell))$ for each ℓ and also on $\overset{(p(\ell+1)-n)}{\wedge} V \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\ell))$. Thus for each $\ell \geq n$ there is an induced action of $\mathrm{Gl}(V)$ on $\mathcal{H}^n_{p,m}$ as an embedded subvariety of the projective space $\mathbb{P}$.

5 Main Results

Our first step will be to identify the semi-stable points for this action of $\mathrm{Gl}(V)$. The main technical tool we will use throughout this section is the following result of Simpson([19]):

Theorem 5.1 (Simpson, [19], Lemma 1.15). *There exists an L such that for $\ell \geq L$ the following holds: Suppose $x : V \otimes O_{\mathbb{P}^1} \to \mathcal{B} \to 0$ is a point in $\mathcal{H}^n_{p,m}$. For any subspace $H \subset V$, let $\mathcal{G}$ denote the subsheaf of $\mathcal{B}$ generated by $H \otimes O_{\mathbb{P}^1}$. Suppose that $\chi(\mathcal{G}, \ell) > 0$ and*

$$\frac{dim\, H}{\chi(\mathcal{G}, \ell)} \leq \frac{dim\, V}{(m(\ell+1)+n)} \tag{5.1}$$

(resp. $<$) for all nonzero proper subspaces $H \subset V$. Then the point x is semi-stable (resp. stable) in the embedding $\rho_\ell : \mathcal{H}^n_{p,m} \to \mathbb{P}^N$ described above.

We want to rephrase this theorem in a geometric form that will be more suitable for our application. As a first step we want to point out that every point in the compactification may be viewed as a map from the projective line $\mathbb{P}^1$ to a Grassmannian.

While this identification is fairly standard for the points $x \in \mathcal{H}^n_{p,m}$ where the quotient sheaf $\mathcal{B}$ is locally free we want to explain how it can be extended to the points where $\mathcal{B}$ is not locally free.

If $x \in \mathcal{H}^n_{p,m}$ corresponds to the short exact sequence (4.2) then $\mathcal{B} \simeq \mathcal{B}_{\text{free}} \oplus \mathcal{B}_{\text{tor}}$. The map ϕ followed by the surjection: $\mathcal{B}_{\text{free}} \oplus \mathcal{B}_{\text{tor}} \to \mathcal{B}_{\text{free}} \to 0$ gives rise to a surjection: $x' : V \otimes O_{\mathbb{P}^1} \xrightarrow{\phi'} \mathcal{B}_{\text{free}}$ which corresponds to an observable system of the same rank, but lower degree than the original system x. The point x' belongs to $\mathcal{H}^{n'}_{p,m}$ where $n' < n$. The system x' is the observable part of the system x.

Let $G' = \mathrm{Grass}(m, V)$ be the Grassmannian of m-dimensional

quotients of V then each point $x \in \mathcal{H}^n_{p,m}$ corresponds to a map $\phi_x : \mathbb{P}^1 \to G'$ given as follows: If x corresponds to the short exact sequence: $0 \to \mathcal{A} \to V \otimes O_{\mathbb{P}^1} \xrightarrow{\phi} \mathcal{B} \to 0$, then we have a surjection ϕ (or ϕ') from $V \otimes O_{\mathbb{P}^1}$ to $\mathcal{B}_{\text{free}}$. This map can be represented by a $m \times (m+p)$ matrix $Q(s,t)$ where each row of this matrix consists of homogeneous polynomials $g(s,t)$ of the same degree. Further at each point $(s,t) \in \mathbb{P}^1$ the matrix $Q(s,t)$ has full row rank. Thus given a point $z \in \mathbb{P}^1$ we can define a quotient denoted by $\phi(z)$ of rank m of the vector space V given by the matrix $Q(s,t)$ evaluated at the point z. We denote the corresponding map from $\mathbb{P}^1$ to G' by ϕ.

Lemma 5.2. *Let $x \in \mathcal{H}^n_{p,m}$. Suppose that for a generic point $z \in \mathbb{P}^1$ and any proper subspace $H \subset V$*

$$\dim(\phi_x(z))(H) > \frac{m}{m+p} \dim H \tag{5.2}$$

where ϕ_x is the map from $\mathbb{P}^1$ to the Grassmannian G' of quotients of V, associated to the point x and $\phi_x(z)(H)$ denotes the image of H under the canonical projection map $V \to \phi_x(z)$. Then there exists an L such that for $\ell \geq L$, x is a stable point in $\rho_\ell(\mathcal{H}^n_{p,m})$.

Proof. In the first part of the proof we will assume that the point x is observable, that is, the sheaf $\mathcal{B}$ is locally free.

Let $H \subset V$ be a proper subspace of V and let the image of H under the map ϕ be $\mathcal{G}$. If (5.2) is satisfied then the rank g of the sheaf $\mathcal{G}$, which is equal to the dimension of $(\phi_x(z))(H)$ at the generic point is greater than $\frac{m}{m+p} \dim H$. Now the Euler characteristic $\chi(\mathcal{G}(\ell)) = g(\ell+1) + \deg \mathcal{G}$. So for ℓ large enough

$$\begin{array}{rcl} \chi(\mathcal{G}(\ell)) & > & \dim H \frac{m}{m+p}(\ell+1) + n \\ & > & \frac{\dim H}{m+p}(m(\ell+1)+n) \\ \text{so } \frac{\dim H}{\chi(\mathcal{G}(\ell))} & < & \frac{m+p}{m(\ell+1)+n}. \end{array}$$

Therefore by Theorem 5.1 the point x is stable.

If x is not observable our assumption is that the map $\phi'_{x'}$ satisfies condition (5.2). The rank of $\phi'_{x'}$ is the rank of the sub-sheaf $\mathcal{G}'$ of $\mathcal{B}_{\text{free}}$ generated by $H \otimes O_{\mathbb{P}^1}$. The sheaf $\mathcal{G}'$ is a sub-sheaf of the sheaf $\mathcal{G}$ which is the sub-sheaf of $\mathcal{B}$ generated by $H \otimes O_{\mathbb{P}^1}$, so $\chi(\mathcal{G}(\ell)) \geq \chi(\mathcal{G}'(\ell))$. By the above calculations $\chi(\mathcal{G}'(\ell))$ satisfies condition 5.1 so $\mathcal{G}$ satisfies this condition as well. Thus the point x is stable. $\square$

We wish to recall the following definition from the systems theory literature:

Definition 5.3. A $p \times (m+p)$ homogeneous autoregressive system $P(s,t)$ is called *nondegenerate* if there is no full rank $m \times (m+p)$ matrix K with entries in the complex numbers $\mathbb{C}$ such that

$$\det \begin{pmatrix} P(s,t) \\ K \end{pmatrix} = 0.$$

One verifies that nondegenerate systems cannot exist if the McMillan degree is 'small'. The following result shows when nondegenerate systems are open and dense inside the variety $\mathcal{H}^n_{p,m}$.

Theorem 5.4 ([1]). *If the McMillan degree $n \geq mp$ then the variety $\mathcal{H}^n_{p,m}$ contains a nonempty Zariski open set of nondegenerate systems.*

Lemma 5.5. *If $P(s,t)$ is a homogeneous autoregressive system that is nondegenerate then the corresponding point $x \in \mathcal{H}^n_{p,m}$ satisfies the condition (5.2) of Lemma 5.2 and is therefore a stable point in $\rho_\ell(\mathcal{H}^n_{p,m})$ for large enough ℓ.*

Proof. Suppose the point $x \in \mathcal{H}^n_{p,m}$ corresponds to the short exact sequence (4.2). Then the map ψ is represented by the transpose of the matrix $P(s,t)$. For each point $z \in \mathbb{P}^1$, $\psi(z)$ determines a subspace of V given by the row span of the matrix $P(s,t)$ evaluated at the point z. If $P(s,t)$ is nondegenerate then for the generic point $z \in \mathbb{P}^1$ and a subspace $H \subset V$, $\dim(H \cap \psi(z)) = 0$ if $\dim H \leq m$ and if $\dim H > m$ then $\dim(H \cap \psi(z)) = \dim H - m$. Notice that $\psi(z)$ is the kernel of the map $\phi(z)$. So if x is a nondegenerate point at the generic point $z \in \mathbb{P}^1$, if $\dim H \leq m$ then $\dim \phi_x(H) = \dim H$ and if $\dim H > m$ then $\dim(\phi_x(H)) = m$. Thus the point x satisfies the condition (5.2) and therefore it is stable. $\square$

We want to remark that the converse of the statement in the above Lemma is not true. One can find stable points in $\mathcal{H}^n_{p,m}$ that are not nondegenerate as the following example shows.

Example 5.6. Let $P(s,t)$ be given by the following matrix:

$$P(s,t) = \begin{pmatrix} s^2 & st & t^2 & s^2 & s^2+t^2 \\ st & t^2 & s^2 & s^2+2t^2 & s^2-t^2 \end{pmatrix} \tag{5.3}$$

where the transpose of $P(s,t)$ is the matrix of the sheaf map ψ in (4.2). Let $x \in \mathcal{H}^4_{3,2}$ be the point represented by the homogeneous autoregeressive system $P(s,t)$. This point is degenerate since

$$\det \begin{pmatrix} & & P(s,t) & & \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 0.$$

The matrix of the map $\phi(s,t)$ in the short exact sequence (4.2) corresponding to this point x was computed to be the following matrix:

$$Q(s,t) = \begin{pmatrix} -t & s & 0 & 0 & 0 \\ s+5t & -2t & s+4t & -2s-t & s-4t \\ -s^2-4st-8t^2 & 3t^2 & -s^2-3st-7t^2 & s^2+4st+2t^2 & 7t^2 \end{pmatrix}.$$

In order to check that the point x is stable, according to Lemma 5.2 it suffices to check that for a generic point $z \in \mathbb{P}^1$, for any subspace H of dimension three the image $\phi_x(z)(H)$ has dimension at least two, and if $\dim H$ is four then the image has dimension at least three. Let $A \in Gl_5$ be a generic matrix. Now, it is enough to check that the first three columns of the matrix QA has rank at least two, and the first four columns has a rank of at least three, at the generic point in $\mathbb{P}^1$. We confirmed this to be the case using the computer algebra system Maculay 2, even though the second computation took more than an hour on a Macintosh G4. Thus the point x is degenerate but stable.

An immediate consequence of Lemma 5.5 is the following theorem:

Theorem 5.7. *The set of semi-stable orbits $X^{ss} \subset \mathcal{H}^n_{p,m}$ contains the set of nondegenerate systems. In particular if $n \geq mp$ then X^{ss} is nonempty and there exists a projective variety Y and a morphism $\phi : X^{ss} \to Y$ having the property that $\phi^{-1}(y)$ contains exactly one closed Gl_{m+p} orbit for every $y \in Y$.*

Proof. The stable orbits (and nondegenerate systems are stable by Lemma 5.5) form a subset of the semi-stable orbits. □

We want to remark that since the stabilizer group of a stable point is a finite group (see [14]) the stabilizer group of a nondegenerate point is finite.

Corollary 5.8. *Consider the set of m input, p output systems of McMillan degree n as introduced in* (1.1). *If $n \geq mp$ then the set of feedback orbits with respect to the full feedback group has a continuous set of invariants consisting of a quasi-projective variety.*

Proof. Consider the set $\mathcal{R}_{n,m,p}$ of proper $p \times m$

transfer functions of McMillan degree n. Let $V := \mathcal{R}_{n,m,p} \cap X^{ss}$ and let $\hat{\phi}$ be the restriction of the morphism ϕ to the Zariski open subset V. Then $\mathrm{im}(\hat{\phi})$ describes a quasi-projective variety parameterizing the orbits under the extended full feedback group as described in Section 2. By Lemma 2.1 this variety also parameterizes the set of m input, p output systems of McMillan degree n as introduced in (1.1) modulo the full feedback group. ☐

6 The feedback orbit classification problem in terms of generalized first order systems

The set of homogeneous autoregressive systems can be described through generalized first order systems and we refer to [15, 18]. In this section we describe the extended feedback group in terms of these generalized first order system. In this way we make the connection with work of Hinrichsen and O'Halloran [9].

Following the exposition in [10, 16] consider $(n+p) \times n$ matrices K, L and a $(n+p) \times (m+p)$ matrix M. Those matrices define a generalized state space system through

$$K\dot{x}(t) + Lx(t) + Mw(t) = 0, \qquad x(t) \in \mathbb{R}^n, w(t) \in \mathbb{R}^{m+p}. \tag{6.1}$$

The system (6.1) is called admissible if the homogeneous pencil $[sK + tL]$ has generically full column rank. An admissible system is called controllable if the pencil $[sK + tL\ M]$ has full row rank for all $(s,t) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

There is a natural equivalence relation among generalized first order systems: If $U \in Gl_{n+p}$ and $S \in Gl_n$ then

$$(K, L, M) \ \sim \ (UKS^{-1}, ULS^{-1}, UM). \tag{6.2}$$

If the high order coefficient matrix

$$[K\, M]$$

has the property that the first $(n+p)\times(n+p)$ minor is invertible then the system (6.1) is equivalent to a system having

$$K = \begin{bmatrix} -I \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} A \\ C \end{bmatrix}, \quad M = \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}, \tag{6.3}$$

i.e. the system is equivalent to a usual state space system of the form

$$\dot{x} = Ax + Bu, \; y = Cx + Du.$$

The connection to the set of homogeneous autoregressive systems is established through:

Theorem 6.1. *([16]) The categorical quotient of the set of controllable state space systems as introduced in* (6.1) *under the group action $Gl_{n+p} \times Gl_n$ is isomorphic to the smooth projective variety $\mathcal{H}^n_{m,p}$ of all $m \times (m+p)$ homogeneous autoregressive systems of McMillan degree n.*

Next we define:

Definition 6.2. Two generalized first order systems are equivalent under the extended full feedback group if there are invertible matrices $S \in Gl_n$, $T \in Gl_{m+p}$ and $U \in Gl_{n+p}$ such that

$$(K, L, M) \;\; \sim \;\; (UKS^{-1}, ULS^{-1}, UMT^{-1}). \tag{6.4}$$

Note that the linear transformation T introduced in (2.3) corresponds to a change of basis in the set of external variables $w = \begin{bmatrix} u \\ y \end{bmatrix}$ and it is therefore equal to the transformation T appearing in (6.4). Finally the group action described in (6.4) corresponds exactly to the transformations $(i), (ii), (iii)$ considered by Hinrichsen and O'Halloran in [9, p. 2730].

As a consequence of Theorem 6.1 we have:

Theorem 6.3. *The categorical quotient induced by the group action* (6.4) *is equal to the projective variety $\mathcal{H}^n_{m,p}/Gl_{m+p}$. Moreover $\mathcal{H}^n_{m,p}/Gl_{m+p}$ represents a continuous parameterization of the feedback orbits under the full feedback group.*

7 A concrete description of the moduli space in the MISO situation

In this section we explain our result in the multi input, single output (i.e. $p = 1$) situation. The variety $\mathcal{H}^n_{1,m}$ consists in this case of all $1 \times (m+1)$ polynomial vectors

$$P(s) = (f_1(s), \ldots, f_{m+1}(s))$$

whose polynomial entries have degree at most n. In this way we can identify the space $\mathcal{H}^n_{1,m}$ with the projective space $\mathbb{P}^N$, where $N = mn + m + n$.

By definition a systems is nondegenerate if the $m+1$ polynomial vectors

$$\{f_1(s), \ldots, f_{m+1}(s)\} \subset \mathbb{R}[s]$$

are linearly independent over $\mathbb{R}$. Clearly this can only happen if the McMillan degree $n \geq m$. Theorem 5.7 and Corollary 5.8 state in this case that the set of systems where $\{f_1(s), \ldots, f_{m+1}(s)\}$ are linearly independent are all contained in the semi-stable orbits and that there is a quasi-projective variety describing the quotient. In our situation this can be made very concrete:

Identify the set of polynomial vectors of degree at most n with the vector space $\mathbb{R}^{n+1}$. A system $P(s) = (f_1(s), \ldots, f_{m+1}(s))$ then defines a linear subspace

$$\mathrm{span}_{\mathbb{R}}\{f_1(s), \ldots, f_{m+1}(s)\} \subset \mathbb{R}^{n+1}.$$

This subspace has dimension $m+1$ if and only if $P(s)$ describes a nondegenerate system. The semi-stable points under the extended full feedback group Gl_{m+1} therefore describe a well defined $m+1$ dimensional subspace of $\mathbb{R}^{n+1}$. The categorical quotient $\mathcal{H}^n_{1,m}/Gl_{m+1}$ is in this case exactly the Grassmann variety $\mathrm{Grass}(m+1, \mathbb{R}^{n+1})$ of $m+1$ dimensional subspaces in $\mathbb{R}^{n+1}$. This extends a construction in [3,6] for output feedback invariants of SISO systems, where $m = 1$.

In particular it follows that the semi-stable orbits coincide exactly with the set of nondegenerate systems, something which is not true for general multi-output systems, as illustrated by Example 5.6.

The case $p = 0$ is interesting and nontrivial as well. In fact, the smooth moduli space, as constructed above, parametrizes representations for the wild input-state quiver action $(K, L, M) \mapsto (UKS^{-1}, ULS^{-1}, UMT^{-1})$. The task of classifying representations of this quiver has been an open problem for at least the last two decades.

8 Conclusion

In this paper we did show that the output feedback problem is closely related to the study of the moduli space $\mathcal{H}^n_{p,m}/Gl_{m+p}$, where $\mathcal{H}^n_{p,m}$ denotes the set of $p \times (m+p)$ homogeneous autoregressive systems of McMillan degree n.

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